4.7 Brownian Bridge

黃于珊

4.7.1 Gaussian Process

Definition 4.7.1:

A Gaussian process $X(t)$, $t \geq 0$, is a stochastic process that has the property that, for arbitrary times $0 < t_1 < t_2 < \ldots < t_n$, the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly normally distributed.

- The joint normal distribution of a set of vectors is determined by their means and covariances.

More generally, a random column vector $\mathbf{X} = (X_1, \ldots, X_n)^{\mathrm{tr}}$, *where the superscript* tr *denotes transpose, is* jointly normal *if it has joint density*

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)C^{-1}(\mathbf{x}-\mu)^{\mathrm{tr}}\right\}. \qquad (2.2.18)$$

In equation (2.2.18), $\mathbf{x} = (x_1, \ldots, x_n)$ *is a row vector of dummy variables,* $\mu = (\mu_1, \ldots, \mu_n)$ *is the row vector of expectations, and* C *is the positive definite matrix of covariances.*

- For a Gaussian process, the joint distribution of $X(t_1), X(t_2), \ldots, X(t_n)$ is determined by the means and covariances of these random variables.
- We denote the mean of $X(t)$ by $m(t)$, and, for $s \geq 0$, $t \geq 0$, we denote the covariance of $X(s)$ and $X(t)$ by $c(s, t)$; i.e.,

$$m(t) = EX(t), c(s,t) = E\left[\left(X(s) - m(s)\right)\left(X(t) - m(t)\right)\right]$$

Example 4.7.2 (Brownian motion)

Brownian motion *W(t)* is a Gaussian process.

For $0 < t_1 < t_2 < \ldots < t_n$, the increments

$$I_1 = W(t_1), I_2 = W(t_2) - W(t_1), \ldots, I_n = W(t_n) - W(t_{n-1})$$

$W(t_0)=0$

are independent and normally distributed.

Increments over nonoverlapping time intervals are independent

$W(t)-W(s) \sim N(0,t-s)$

Writing

$W(t_2)=W(t_1)+I_2$

$$W(t_1) = I_1, W(t_2) = \sum_{j=1}^{2} I_j, \ldots, W(t_n) = \sum_{j=1}^{n} I_j,$$

Example 4.7.2 (Brownian motion)

$0 < t_1 < t_2 < \ldots < t_n$

- The random variables $W(t_1), W(t_2), \ldots, W(t_n)$ are jointly normally distributed. Gaussian process

Independent normal random variables are jointly normal.

➡ $I_1, I_2, \ldots, I_n$ are jointly normal

Linear combinations of jointly normal random variables are jointly normal.

- The mean function for Brownian motion is

$$m(t) = EW(t) = 0$$

Example 4.7.2 (Brownian motion)

- We may compute the covariance by letting $0 \le s \le t$ be given and noting that

E[W(s)W(t)]-E[W(s)]E[W(t)]

$$c(s,t) = E\left[W(s)W(t)\right] = E\left[W(s)\left(W(t) - W(s) + W(s)\right)\right]$$
$$= E\left[W(s)\left(W(t) - W(s)\right)\right] + E\left[W^2(s)\right]$$

- Because *W(s)* and *W(t)* - *W(s)* are independent and both have mean zero, we see that

$$E\left[W(s)\left(W(t) - W(s)\right)\right] = 0$$

Example 4.7.2 (Brownian motion)

$$c(s,t) = E[W(s)W(t)] = E[W(s)(W(t) - W(s) + W(s))]$$
$$= E[W(s)(W(t) - W(s))] + E[W^2(s)]$$

- The other term, $E[W^2(s)]$, is the variance of $W(s)$, which is s.

$E[W^2(s)]-(E[W(s)])^2$

$$\mathrm{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i. \quad (3.3.3)$$

Example 4.7.2 (Brownian motion)

- We conclude that $c(s,t)=s$ when $0 \le s \le t$.
- Reversing the roles of s and t, we conclude that $c(s,t)=t$ when $0 \le t \le s$.
- In general, the covariance function for Brownian motion is then

$$c(s,t) = s \wedge t,$$

where $s \wedge t$ denotes the minimum of s and t.

Example 4.7.3
(Itô integral of a deterministic integrand)

- Let $\Delta(t)$ be a nonrandom function of time, and define
$$I(t) = \int_0^t \Delta(s)\,dW(s)$$
where $W(t)$ is a Brownian motion. Then $I(t)$ is a Gaussian process, as we now show.
- In the proof of Theorem 4.4.9, we showed that, for fixed $u \in R$, the process
$$M_u(t) = \exp\left\{uI(t) - \frac{1}{2}u^2 \int_0^t \Delta^2(s)\,ds\right\}$$
is a martingale.

Example 4.7.3
(Itô integral of a deterministic integrand)

- Then $M_u(t) = \exp\left\{ uI(t) - \frac{1}{2}u^2 \int_0^t \Delta^2(s)ds \right\}$ is a martingale.

$$1 = M_u(0) = EM_u(t) = e^{-\frac{1}{2}u^2 \int_0^t \Delta^2(s)ds} \cdot Ee^{uI(t)}$$

and we thus obtained the moment-generating function formula

$$Ee^{uI(t)} = e^{\frac{1}{2}u^2 \int_0^t \Delta^2(s)ds}$$

$X \sim \mathcal{N}(\mu, \sigma^2)$

$M_X(t) = \mathbb{E}(e^{tX})$

$= e^{t\mu + t^2\sigma^2/2}$

(with mean zero and variance $\int_0^t \Delta^2(s)ds$)

- I(t)~N(0, $\int_0^t \Delta^2(s)ds$)

Example 4.7.3
(Itô integral of a deterministic integrand)

- We have shown that $I(t)$ is normally distributed, verification that the process is Gaussian requires more.
- Verify that, for $0 < t_1 < t_2 < \ldots < t_n$, the random variables $I(t_1), I(t_2), \ldots, I(t_n)$ are jointly normally distributed.

Example 4.7.3
(Itô integral of a deterministic integrand)

$I(t) \sim N(0, \int_0^t \Delta^2(s)ds)$

- It turns out that the increments

$$I(t_1) - I(0) = I(t_1), I(t_2) - I(t_1), \ldots, I(t_n) - I(t_{n-1})$$

are **normally distributed** and **independent**(p.14), and from this the joint normality of $I(t_1), I(t_2), \ldots, I(t_n)$ follows by the same argument as used in Example 4.7.2 for Brownian motion.

$I(t_1)=(I(t_1)-I(0))$

$I(t_2)=(I(t_2)-I(t_1))+(I(t_1)-I(0))$

Independent normal random variables are jointly normal.

➡ $(I(t_1)-I(0)), (I(t_2)-I(t_1)), \ldots, (I(t_n)-I(t_{n-1}))$ are jointly normal

Linear combinations of jointly normal random variables are jointly normal.

Example 4.7.3
(Itô integral of a deterministic integrand)

- Next, we show that, for $0 < t_1 < t_2$, the two random increments $I(t_1)-I(0)=I(t_1)$ and $I(t_2)-I(t_1)$ are normally distributed and independent.
- The argument we provide can be iterated to prove this result for any number of increments.

Example 4.7.3
(Itô integral of a deterministic integrand)

- For fixed $u_2 \in R$, the martingale property of M_{u_2} implies that

$$M_{u_2}(t_1) = E[M_{u_2}(t_2) \mid F(t_1)]$$

- Now let $u_1 \in R$ be fixed. Because $\frac{M_{u_1}(t_1)}{M_{u_2}(t_1)}$ is $F(t_1)$-measurable, we may multiply the equation above by this quotient to obtain

$$M_{u_1}(t_1) = E\left[\frac{M_{u_1}(t_1) M_{u_2}(t_2)}{M_{u_2}(t_1)} \mid F(t_1)\right]$$

$$M_u(t) = \exp\left\{uI(t) - \frac{1}{2}u^2 \int_0^t \Delta^2(s)\,ds\right\}$$

$$= E\left[\exp\left\{u_1 I(t_1) + u_2(I(t_2) - I(t_1)) - \frac{1}{2}u_1^2 \int_0^{t_1} \Delta^2(s)\,ds - \frac{1}{2}u_2^2 \int_{t_1}^{t_2} \Delta^2(s)\,ds\right\} \mid F(t_1)\right]$$

Example 4.7.3
(Itô integral of a deterministic integrand)

$$M_{u_1}(t_1) = E\left[\exp\left\{u_1 I(t_1) + u_2(I(t_2) - I(t_1)) - \frac{1}{2}u_1^2\int_0^{t_1}\Delta^2(s)ds - \frac{1}{2}u_2^2\int_{t_1}^{t_2}\Delta^2(s)ds\right\} \mid F(t_1)\right]$$

- We now take expectations

$$M_u(t) = \exp\left\{uI(t) - \frac{1}{2}u^2\int_0^t \Delta^2(s)ds\right\}$$

$$\mathbb{E}(\mathbb{E}[X|\mathcal{G}]) = \mathbb{E}X$$

$$1 = M_{u_1}(0) = EM_{u_1}(t_1)$$

$$= E\left[\exp\left\{u_1 I(t_1) + u_2(I(t_2) - I(t_1)) - \frac{1}{2}u_1^2\int_0^{t_1}\Delta^2(s)ds - \frac{1}{2}u_2^2\int_{t_1}^{t_2}\Delta^2(s)ds\right\}\right]$$

$$= E\left[\exp\{u_1 I(t_1) + u_2(I(t_2) - I(t_1))\}\right]\cdot\exp\left\{-\frac{1}{2}u_1^2\int_0^{t_1}\Delta^2(s)ds - \frac{1}{2}u_2^2\int_{t_1}^{t_2}\Delta^2(s)ds\right\}$$

- Where we have used the fact that $\Delta^2(s)$ is nonrandom to take the integrals of $\Delta^2(s)$ outside the expectation on the right-hand side.

Example 4.7.3
(Itô integral of a deterministic integrand)

$$1 = E\left[\exp\{u_1 I(t_1) + u_2(I(t_2) - I(t_1))\}\right] \cdot \exp\left\{-\frac{1}{2}u_1^2 \int_0^{t_1} \Delta^2(s)ds - \frac{1}{2}u_2^2 \int_{t_1}^{t_2} \Delta^2(s)ds\right\}$$

- This leads to the moment-generating function formula

$$E\left[\exp\{u_1 I(t_1) + u_2(I(t_2) - I(t_1))\}\right]$$
$$= \exp\left\{\frac{1}{2}u_1^2 \int_0^{t_1} \Delta^2(s)ds\right\} \cdot \exp\left\{\frac{1}{2}u_2^2 \int_{t_1}^{t_2} \Delta^2(s)ds\right\}$$

Example 4.7.3
(Itô integral of a deterministic integrand)

$$E\left[\exp\left\{u_1 I(t_1) + u_2 (I(t_2) - I(t_1))\right\}\right]$$
$$= \exp\left\{\frac{1}{2} u_1^2 \int_0^{t_1} \Delta^2(s)ds\right\} \cdot \exp\left\{\frac{1}{2} u_2^2 \int_{t_1}^{t_2} \Delta^2(s)ds\right\}$$

- The right hand side is the product of
  - the moment-generating function for a normal random variable with mean zero and variance $\int_0^{t_1} \Delta^2(s)ds$
  - the moment-generating function for a normal random variable with mean zero and variance $\int_{t_1}^{t_2} \Delta^2(s)ds$

$$X \sim \mathcal{N}(\mu, \sigma^2)$$
$$M_X(t) = \mathbb{E}(e^{tX})$$
$$= e^{t\mu + t^2\sigma^2/2}$$

Example 4.7.3
(Itô integral of a deterministic integrand)

$$E\left[\exp\{u_1 I(t_1) + u_2 (I(t_2) - I(t_1))\}\right]$$

$$= \exp\left\{\frac{1}{2}u_1^2 \int_0^{t_1} \Delta^2(s)ds\right\} \cdot \exp\left\{\frac{1}{2}u_2^2 \int_{t_1}^{t_2} \Delta^2(s)ds\right\}$$

$I(t) \sim N(0, \int_0^t \Delta^2(s)ds)$

$I(t_1) \sim N(0, \int_0^{t_1} \Delta^2(s)ds)$ $\quad I(t_2) - I(t_1) \sim N(0, \int_{t_1}^{t_2} \Delta^2(s)ds)$

- It follows that $I(t_1)$ and $I(t_2)-I(t_1)$ must have these distributions, and because their joint moment-generating function factors into this **product of moment-generating functions**, they must be independent.

Example 4.7.3
(Itô integral of a deterministic integrand)

E[I(t₁)I(t₂)]-E[I(t₁)]E[I(t₂)]

- We have $c(t_1,t_2) = E\big[I(t_1)I(t_2)\big] = E\big[I(t_1)\big(I(t_2) - I(t_1) + I(t_1)\big)\big]$

$$= E\big[I(t_1)\big(I(t_2) - I(t_1)\big)\big] + EI^2(t_1)$$

$$\mathbb{E}I^2(t) = \int_0^t \Delta^2(s)\,ds$$

I(t₁)和I(t₂)-I(t₁)獨立

$$= EI(t_1)\cdot E\big[I(t_2) - I(t_1)\big] + \int_0^{t_1} \Delta^2(s)ds$$

$$= \int_0^{t_1} \Delta^2(s)ds$$

$$0 < t_1 < t_2$$

- For the general case where s ≥ 0 and t ≥ 0 and we do not know the relationship between s and t, we have the covariance formula

$$c(s,t) = \int_0^{s\wedge t} \Delta^2(u)du$$

4.7.2 Brownian Bridge as a Gaussian Process

Definition 4.7.4.

Let $W(t)$ be a Brownian motion. Fix $T>0$. We define the Brownian bridge from 0 to 0 (p.22) on $[0,T]$ to be the process

$$X(t) = W(t) - \frac{t}{T}W(T), 0 \le t \le T \qquad (4.7.2)$$

4.7.2 Brownian Bridge as a Gaussian Process

$$X(t) = W(t) - \frac{t}{T} W(T), 0 \leq t \leq T$$

- The process $X(t)$ satisfies

$$X(0) = X(T) = 0$$

t=0 X(0)=W(0)-0=0
t=T X(T)=W(T)-W(T)=0

- Because $W(T)$ enters the definition of $X(t)$ for $0 \leq t \leq T$, the Brownian bridge $X(t)$ is not adapted to the filtration F(t) generated by W(t).

4.7.2 Brownian Bridge as a Gaussian Process

$$X(t) = W(t) - \frac{t}{T} W(T), 0 \le t \le T$$

- For $0 < t_1 < t_2 < \ldots < t_n < T$, the random variables

$$X(t_1) = W(t_1) - \frac{t_1}{T} W(T), \ldots, X(t_n) = W(t_n) - \frac{t_n}{T} W(T)$$

are jointly normal because $W(t_1), \ldots, W(t_n)$, $W(T)$ are jointly normal.

p.6 Example 4.7.2
Linear combinations of jointly normal random variables are jointly normal.

- Hence, the Brownian bridge from 0 to 0 is a Gaussian process.

4.7.2 Brownian Bridge as a Gaussian Process

- Its mean function is easily seen to be

$$X(t) = W(t) - \frac{t}{T}W(T)$$

$$\text{E}[W(t)] - \frac{t}{T}E[W(T)]$$

$$m(t) = EX(t) = E\left[W(t) - \frac{t}{T}W(T)\right] = 0$$

- For $s,t \in (0,T)$,we compute the covariance function E[X(s)X(t)]-E[X(s)]E[X(t)]

X(s) X(t)

$$c(s,t) = E\left[\left(W(s) - \frac{s}{T}W(T)\right)\left(W(t) - \frac{t}{T}W(T)\right)\right]$$

$$E[W(s)W(t)] = s \wedge t$$

$$= E[W(s)W(t)] - \frac{t}{T}E[W(s)W(T)] - \frac{s}{T}E[W(t)W(T)] + \frac{st}{T^2}EW^2(T)$$

$$= s \wedge t - \frac{2st}{T} + \frac{st}{T} = s \wedge t - \frac{st}{T}$$

4.7.2 Brownian Bridge as a Gaussian Process

Definition 4.7.5.

Let $W(t)$ be a Brownian motion. Fix $T>0$, $a \in R$ and $b \in R$. We define the Brownian bridge from a to b on $[0, T]$ to be the process

Gaussian process — Gaussian process

$$\underline{X^{a\to b}(t)} = a + \frac{(b-a)t}{T} + \underline{X(t)}, 0 \le t \le T$$

where $X(t) = X^{0\to 0}$ is the Brownian bridge from 0 to 0.

Begins at a at time 0 and ends at b at time T.

t=0 $X^{a\to b}(0) = a+0+X(0)=a$

t=T $X^{a\to b}(T) = a+(b-a)+X(T)=b$

$$\boxed{X(0) = X(T) = 0}$$

Adding a nonrandom function to a Gaussian process gives us another Gaussian process.

4.7.2 Brownian Bridge as a Gaussian Process

- The mean function is affected: $X^{a\to b}(t) = a + \frac{(b-a)t}{T} + X(t)$

$$m^{a\to b}(t) = EX^{a\to b}(t) = a + \frac{(b-a)t}{T} \quad \boxed{EX(t) = 0}$$

- However, the covariance function is not affected:

$$c^{a\to b}(s,t) = E\left[\left(X^{a\to b}(s) - m^{a\to b}(s)\right)\left(X^{a\to b}(t) - m^{a\to b}(t)\right)\right] = s \wedge t - \frac{st}{T}$$

$$=E[((a+\frac{(b-a)s}{T} + X(s))-(a+\frac{(b-a)s}{T}))((a+\frac{(b-a)t}{T} + X(t))-(a+\frac{(b-a)t}{T}))]$$

$$=E[X(s)X(t)]$$

$$E[X(s)X(t)] = s \wedge t - \frac{st}{T}$$

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

- We cannot write the Brownian bridge as a stochastic integral of a deterministic integrand because the variance of the Brownian bridge,

$EX(t) = 0$

$$EX^2(t) = c(t,t) = t - \frac{t^2}{T} = \frac{t(T-t)}{T}$$

X(s) X(t)

$c(s,t) = s \wedge t - \frac{st}{T}$

increases for $0 \le t \le T/2$ and then decreases for $T/2 \le t \le T$.

- In Example 4.7.3, the variance of $I(t) = \int_0^t \Delta(u)dW(u)$ is $\int_0^t \Delta^2(u)du$, which is nondecreasing in t.

F.O.C 1- $\frac{2t}{T} = 0$

t=$\frac{T}{2}$

S.O.C - $\frac{2}{T} < 0$

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

- We can obtain a process with the same distribution as the Brownian bridge from 0 to 0 as a scaled stochastic integral.

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

- Consider

$$Y(t) = (T-t)\int_0^t \frac{1}{T-u}\,dW(u), 0 \le t < T$$

p.31

- The integral $I(t) = \int_0^t \frac{1}{T-u}\,dW(u)$

  is a Gaussian process of the type discussed in Example 4.7.3, provided $t < T$ so the integrand is defined.

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

- For $0 < t_1 < t_2 < \ldots < t_n < T$, the random variables

  $Y(t_1) = (T - t_1)I(t_1),\ Y(t_2) = (T - t_2)I(t_2), \ldots, Y(t_n) = (T - t_n)I(t_n)$

  are jointly normal because $I(t_1), I(t_2), \ldots, I(t_n)$ are jointly normal.
- In particular, Y is a Gaussian process.

$$Y(t) = (T - t)\int_0^t \frac{1}{T-u}\,dW(u), 0 \le t < T \qquad I(t) = \int_0^t \frac{1}{T-u}\,dW(u)$$

Linear combinations of jointly normal random variables are jointly normal.

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

- The mean and covariance functions of I are

$m^I(t) = 0$

$$\left[\frac{1}{T-u}\right]_0^{s\wedge t} \qquad \begin{matrix} \text{V=T-u} & \int -V^{-2}dV \\ \text{dV=-du} & [V^{-1}] \end{matrix}$$

$$c^I(s,t) = \int_0^{s\wedge t} \frac{1}{(T-u)^2}\,du = \frac{1}{T-s\wedge t} - \frac{1}{T} \text{ for all } s,t \in [0,T)$$

- This means that the mean function for *Y* is

$m^Y(t) = 0$

$$\text{I(t)} \sim \text{N}\left(0, \int_0^t \Delta^2(s)ds\right) \quad c(s,t) = \int_0^{s\wedge t} \Delta^2(u)du$$

$$I(t) = \int_0^t \frac{1}{T-u}\,dW(u) \qquad Y(t) = (T-t)\int_0^t \frac{1}{T-u}\,dW(u)$$

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

- To compute the covariance function for Y, we assume for the moment that $0 \le s \le t \le T$ so that

$$c^I(s,t) = \frac{1}{T-s} - \frac{1}{T} = \frac{s}{T(T-s)}$$

$$c^I(s,t) = \frac{1}{T - s \wedge t} - \frac{1}{T}$$

Then

E[Y(s)Y(t)]-E[Y(s)]E[Y(t)]

$$c^Y(s,t) = E[(T-s)(T-t)I(s)I(t)] = (T-s)(T-t)\frac{s}{T(T-s)} = \frac{(T-t)s}{T} = s - \frac{st}{T}$$

- If we had taken $0 \le t \le s < T$, the roles of s and t would have be reversed. In general

$$c^Y(s,t) = s \wedge t - \frac{st}{T}, \forall s,t \in [0,T)$$

$$Y(t) = (T-t)\int_0^t \frac{1}{T-u}\,dW(u)$$

$$I(t) = \int_0^t \frac{1}{T-u}\,dW(u)$$

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

- This is the same covariance formula (4.7.3) we obtained for the Brownian bridge.
- Because the mean and covariance functions for Gaussian process completely determine the distribution of the process, we conclude that the process *Y* has the same distribution as the Brownian bridge from 0 to 0 on [0,T].

$m(t) = EX(t) = 0$	$m^Y(t) = 0$
X(s) X(t) $c(s,t) = s \wedge t - \frac{st}{T}$	$c^Y(s,t) = s \wedge t - \frac{st}{T}$

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

- We now consider the variance

$$c^Y(s,t) = s \wedge t - \frac{st}{T}$$

$$m^Y(t) = 0 \qquad EY^2(t) = c^Y(t,t) = \frac{t(T-t)}{T}, 0 < t < T$$

- Note that, as $t \to T$, this variance converges to 0.
  - As $t \to T$ => the random process $Y(t)$ has mean=0 => variance converges to 0.
- We did not initially define $Y(T)$, but this observation suggests that it makes sense to define $Y(T)=0$.
- If we do that, then $Y(t)$ is continuous at $t=T$.

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

Theorem 4.7.6

Define the process

$$Y(t) = \begin{cases} (T-t)\int_0^t \frac{1}{T-u}\, dW(u) & \text{for } 0 \le t < T, \\ 0 & \text{for } \quad t = T \end{cases}$$

Then $Y(t)$ is a continuous Gaussian process on $[0,T]$ and has mean and covariance functions

$$m^Y(t) = 0, t \in [0,T]$$

$$c^Y(s,t) = s \wedge t - \frac{st}{T}, \forall s,t \in [0,T]$$

In particular, the process $Y(t)$ has the same distribution as the Brownian bridge from 0 to 0 on $[0,T]$ (Definition 4.7.5)

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

- We note that the process $Y(t)$ is adapted to the filtration generated by the Brownian motion $W(t)$.

$$Y(t) = (T - t)\int_0^t \frac{1}{T - u}\, dW(u)$$

- Compute the stochastic differential of $Y(t)$, which is $Y(t) = (T-t)\int_0^t \frac{1}{T-u}\, dW(u)$

$$dY(t) = \int_0^t \frac{1}{T-u}\, dW(u) \cdot d(T-t) + (T-t) \cdot d\int_0^t \frac{1}{T-u}\, dW(u)$$

$$= -\int_0^t \frac{1}{T-u}\, dW(u) \cdot dt + dW(t)$$

$$= -\frac{Y(t)}{T-t}\, dt + dW(t)$$

$$I(t) = \int_0^t \Delta(u)\, dW(u) \quad (4.2.11)$$

$$dI(t) = \Delta(t)\, dW(t) \quad (4.2.12)$$

$$I(t) = \int_0^t \frac{1}{T-u}\, dW(u)$$

$$\frac{1}{\mathrm{T}-t}\, dW(t)$$

4.7.3 Brownian Bridge as a Scaled Stochastic Integral

- If $Y(t)$ is positive as t approaches T, the drift term $-\frac{Y(t)}{T-t}dt$ becomes large in absolute value and is negative.
  - This drives $Y(t)$ toward zero.

$$dY(t) = -\frac{Y(t)}{T-t}dt + dW(t)$$

- On the other hand, if $Y(t)$ is negative, the drift term becomes large and positive, and this again drives $Y(t)$ toward zero.
- This strongly suggests, and it is indeed true, that as $t \longrightarrow T$ the process $Y(t)$ converges to zero almost surely.

4.7.4 Multidimensional Distribution of the Brownian Bridge

- We fix $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and let $X^{a\to b}(t)$ denote the Brownian bridge from a to b on $[0,T]$. We also fix $0 = t_0 < t_1 < t_2 < \ldots < t_n < T$. In this section, We compute the joint density of $X^{a\to b}(t_1), \cdots, X^{a\to b}(t_n)$.

  We recall that the Brownian bridge from a to b has the mean function

$$m^{a\to b}(t) = a + \frac{(b-a)t}{T} = \frac{(T-t)a}{T} + \frac{bt}{T}$$

and covariance function

$$c(s,t) = s \wedge t - \frac{st}{T}$$

When $s \le t$, we may write this as

$$c(s,t) = s - \frac{st}{T} = \frac{s(T-t)}{T} \quad , \quad 0 \le s \le t \le T$$

To simplify notation, we set $\tau_j = T - t_j$ so that $\tau_0 = T$. T- t_0

- We define random variable

$$Z_j = \frac{X^{a\to b}(t_j)}{\tau_j} - \frac{X^{a\to b}(t_{j-1})}{\tau_{j-1}}$$

Because $X^{a\to b}(t_1),\cdots,X^{a\to b}(t_n)$ are jointly normal, so that $Z(t_1),\cdots,Z(t_n)$ are jointly normal. We compute EZ_j , $Var(Z_j)$ and $Cov(Z_i,Z_j)$.

Linear combinations of jointly normal random variables are jointly normal.

$$\tau_j = T - t_j$$

$$E(Z_j) = \frac{1}{\tau_j} EX^{a\to b}(t_j) - \frac{1}{\tau_{j-1}} EX^{a\to b}(t_{j-1}) = \frac{a}{T} + \frac{bt_j}{T\tau_j} - \frac{a}{T} - \frac{bt_{j\text{-}1}}{T\tau_{j\text{-}1}} = \frac{bt_j(T - t_{j-1}) - bt_{j\text{-}1}(T - t_j)}{T\tau_j\tau_{j\text{-}1}}$$

$$= \frac{b(t_j - t_{j-1})}{\tau_j\tau_{j\text{-}1}}.$$

$$\frac{(T - t_j)a}{T} + \frac{bt_j}{T} \qquad \frac{(T - t_{j-1})a}{T} + \frac{bt_{j-1}}{T}$$

$$Z_j = \frac{X^{a\to b}(t_j)}{\tau_j} - \frac{X^{a\to b}(t_{j-1})}{\tau_{j-1}}$$

$$m^{a\to b}(t) = a + \frac{(b-a)t}{T} = \frac{(T-t)a}{T} + \frac{bt}{T}$$

$$Var(Z_j) = \frac{1}{\tau_j^2} Var(X^{a\to b}(t_j)) - \frac{2}{\tau_j \tau_{j-1}} Cov(X^{a\to b}(t_j), X^{a\to b}(t_{j-1})) + \frac{1}{\tau_{j-1}^2} Var(X^{a\to b}(t_{j-1}))$$

$$= \frac{1}{\tau_j^2} c(t_j, t_j) - \frac{2}{\tau_j \tau_{j-1}} c(t_j, t_{j-1}) + \frac{1}{\tau_{j-1}^2} c(t_{j-1}, t_{j-1})$$

$$= \frac{t_j}{T\tau_j} - \frac{2t_{j\text{-}1}}{T\tau_{j\text{-}1}} + \frac{t_{j\text{-}1}}{T\tau_{j\text{-}1}} = \frac{t_j(T - t_{j-1}) - 2t_{j\text{-}1}(T - t_j) + t_{j\text{-}1}(T - t_j)}{T\tau_j \tau_{j\text{-}1}} = \frac{t_j - t_{j-1}}{\tau_j \tau_{j\text{-}1}}.$$

$$Z_j = \frac{X^{a\to b}(t_j)}{\tau_j} - \frac{X^{a\to b}(t_{j-1})}{\tau_{j-1}}$$

$$c(s,t) = s - \frac{st}{T} = \frac{s(T-t)}{T} \quad , \ 0 \le s \le t \le T$$

$$c(t_{j-1}, t_j) = t_{j-1} - \frac{t_{j-1} t_j}{T} = \frac{t_{j-1}(T - t_j)}{T} = \frac{t_{j-1}\tau_j}{T}$$

$$\tau_j = T - t_j$$

$$Cov(Z_i, Z_j) = \frac{1}{\tau_i \tau_j} c(t_i, t_j) - \frac{1}{\tau_i \tau_{j-1}} c(t_i, t_{j-1}) - \frac{1}{\tau_{i-1} \tau_j} c(t_{i-1}, t_j) + \frac{1}{\tau_{i-1} \tau_{j-1}} c(t_{i-1}, t_{j-1})$$

$i < j$

$$= \frac{t_i (T - t_j)}{T \tau_i \tau_j} - \frac{t_i (T - t_{j-1})}{T \tau_i \tau_{j-1}} - \frac{t_{i-1} (T - t_j)}{T \tau_{i-1} \tau_j} + \frac{t_{i-1} (T - t_{j-1})}{T \tau_{i-1} \tau_{j-1}} = 0.$$

$$\tau_j = T - t_j$$

$$Z_j = \frac{X^{a \to b}(t_j)}{\tau_j} - \frac{X^{a \to b}(t_{j-1})}{\tau_{j-1}}$$

$$c(s,t) = s - \frac{st}{T} = \frac{s(T-t)}{T} \quad , \quad 0 \le s \le t \le T$$

$$c(t_{j-1}, t_j) = t_{j-1} - \frac{t_{j-1} t_j}{T} = \frac{t_{j-1}(T - t_j)}{T}$$

- $Z(t_1),\cdots,Z(t_n)$ are jointly normal.
- $Cov(Z_i, Z_j) = 0$.

$$f_{X,Y}(x,y)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}, \qquad (2.2.17)$$

Cov(X,Y)=0 ⇔ X,Y are independent

- The normal random variable $Z_1, \ldots, Z_n$ are independent.

- So we conclude that the normal random variable $Z_1, \ldots, Z_n$ are independent, and we can write down their joint density, which is

$$f_{Z(t_1),\ldots,Z(t_n)}(z_1,\ldots,z_n)=\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi\boxed{\dfrac{t_j-t_{j-1}}{\tau_j\tau_{j\text{-}1}}}}}\exp\left\{-\frac{1}{2}\cdot\frac{\left(z_j-\boxed{\dfrac{b(t_j-t_{j-1})}{\tau_j\tau_{j\text{-}1}}}\right)^2}{\boxed{\dfrac{t_j-t_{j-1}}{\tau_j\tau_{j\text{-}1}}}}\right\}$$

mean

variance　　variance

$$E(Z_j)=\frac{b(t_j-t_{j\text{-}1})}{\tau_j\tau_{j\text{-}1}}$$

$$Var(Z_j)=\frac{t_j-t_{j\text{-}1}}{\tau_j\tau_{j\text{-}1}}$$

$$=\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{\left(z_j-\dfrac{b(t_j-t_{j-1})}{\tau_j\tau_{j\text{-}1}}\right)^2}{\dfrac{t_j-t_{j-1}}{\tau_j\tau_{j\text{-}1}}}\right\}\cdot\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi\dfrac{t_j-t_{j-1}}{\tau_j\tau_{j\text{-}1}}}}.$$

接下來把z的部份做變數變換

we make the change of variables

$$z_j=\frac{x_j}{\tau_j}-\frac{x_{j-1}}{\tau_{j-1}},\qquad j=1,\ldots,n,$$

$X^{a\to b}(0)$=a

- Where $x_0 = a$, to find joint density for $X^{a\to b}(t_1),\cdots,X^{a\to b}(t_n)$. We work first on the sum in the exponent to see the effect of this change of variables.

$$f_{Z(t_1),\ldots,Z(t_n)}(z_1,\ldots,z_n) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi \frac{t_j - t_{j-1}}{\tau_j \tau_{j\text{-}1}}}} \exp\left\{ -\frac{1}{2} \cdot \frac{\left(z_j - \frac{b(t_j - t_{j-1})}{\tau_j \tau_{j\text{-}1}} \right)^2}{\frac{t_j - t_{j-1}}{\tau_j \tau_{j\text{-}1}}} \right\}$$

$$= \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{\left(z_j - \frac{b(t_j - t_{j-1})}{\tau_j \tau_{j\text{-}1}} \right)^2}{\frac{t_j - t_{j-1}}{\tau_j \tau_{j\text{-}1}}} \right\} \cdot \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi \frac{t_j - t_{j-1}}{\tau_j \tau_{j\text{-}1}}}}.$$

- We have

$$\sum_{j=1}^{n} \frac{\left(z_j - \frac{b(t_j - t_{j-1})}{\tau_j \tau_{j-1}}\right)^2}{\frac{t_j - t_{j-1}}{\tau_j \tau_{j-1}}}$$

$$z_j = \frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}}$$

$$= \sum_{j=1}^{n} \frac{\tau_j \tau_{j-1}}{t_j - t_{j-1}} \left(\frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}} - \frac{b(t_j - t_{j-1})}{\tau_j \tau_{j-1}} \right)^2$$

$$= \sum_{j=1}^{n} \frac{\tau_j \tau_{j-1}}{t_j - t_{j-1}} \left(\frac{x_j^2}{\tau_j^2} + \frac{x_{j-1}^2}{\tau_{j-1}^2} + \frac{b^2(t_j - t_{j-1})^2}{\tau_j^2 \tau_{j-1}^2} - \frac{2x_j x_{j-1}}{\tau_j \tau_{j-1}} - \frac{2x_j b(t_j - t_{j-1})}{\tau_j^2 \tau_{j-1}} + \frac{2x_{j-1} b(t_j - t_{j-1})}{\tau_j \tau_{j-1}^2} \right)$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

$$= \sum_{j=1}^{n} \frac{\tau_j \tau_{j-1}}{t_j - t_{j-1}} \left(\frac{x_j^2}{\tau_j^2} + \frac{x_{j-1}^2}{\tau_{j-1}^2} + \frac{b^2(t_j - t_{j-1})^2}{\tau_j^2 \tau_{j-1}^2} - \frac{2x_j x_{j-1}}{\tau_j \tau_{j-1}} \right.$$

$$\left. - \frac{2x_j b(t_j - t_{j-1})}{\tau_j^2 \tau_{j-1}} + \frac{2x_{j-1} b(t_j - t_{j-1})}{\tau_j \tau_{j-1}^2} \right)$$

$$= \sum_{j=1}^{n} \left(\frac{\tau_{j-1} x_j^2}{\tau_j (t_j - t_{j-1})} + \frac{\tau_j x_{j-1}^2}{\tau_{j-1} (t_j - t_{j-1})} + \frac{b^2(t_j - t_{j-1})}{\tau_j \tau_{j-1}} - \frac{2x_j x_{j-1}}{t_j - t_{j-1}} \right.$$

$$\left. - \frac{2x_j b}{\tau_j} + \frac{2x_{j-1} b}{\tau_{j-1}} \right)$$

$$= \sum_{j=1}^{n} \left(\frac{\tau_{j-1} x_j^2}{\tau_j (t_j - t_{j-1})} + \frac{\tau_j x_{j-1}^2}{\tau_{j-1}(t_j - t_{j-1})} + \frac{b^2 (t_j - t_{j-1})}{\tau_j \tau_{j-1}} - \frac{2 x_j x_{j-1}}{t_j - t_{j-1}} - \frac{2 x_j b}{\tau_j} + \frac{2 x_{j-1} b}{\tau_{j-1}} \right)$$

$$= \sum_{j=1}^{n} \left[\frac{x_j^2}{t_j - t_{j-1}} \left(1 + \frac{\tau_{j-1} - \tau_j}{\tau_j} \right) + \frac{x_{j-1}^2}{t_j - t_{j-1}} \left(1 - \frac{\tau_{j-1} - \tau_j}{\tau_{j-1}} \right) - \frac{2 x_j x_{j-1}}{t_j - t_{j-1}} \right] + b^2 \sum_{j=1}^{n} \left(\frac{1}{\tau_j} - \frac{1}{\tau_{j-1}} \right) - 2b \sum_{j=1}^{n} \left(\frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}} \right).$$

$$1 + \frac{\tau_{j-1} - \tau_j}{\tau_j} = \frac{\tau_j + \tau_{j-1} - \tau_j}{\tau_j} = \frac{\tau_{j-1}}{\tau_j}$$

$$1 - \frac{\tau_{j-1} - \tau_j}{\tau_{j-1}} = \frac{\tau_{j-1} - \tau_{j-1} + \tau_j}{\tau_{j-1}} = \frac{\tau_j}{\tau_{j-1}}$$

$$\left(\frac{1}{\tau_j} - \frac{1}{\tau_{j-1}} \right) = \frac{\tau_{j-1} - \tau_j}{\tau_j \tau_{j-1}} = \frac{t_j - t_{j-1}}{\tau_j \tau_{j-1}}$$

$$\tau_{j-1} - \tau_j = (T - t_{j-1}) - (T - t_j) = t_j - t_{j-1}$$

$$\tau_{j-1} - \tau_j = (T - t_{j-1}) - (T - t_j) = t_j - t_{j-1}$$

$$= \sum_{j=1}^{n} \left[\frac{x_j^2}{t_j - t_{j-1}} \left(1 + \frac{\tau_{j-1} - \tau_j}{\tau_j} \right) + \frac{x_{j-1}^2}{t_j - t_{j-1}} \left(1 - \frac{\tau_{j-1} - \tau_j}{\tau_{j-1}} \right) - \frac{2x_j x_{j-1}}{t_j - t_{j-1}} \right] + b^2 \sum_{j=1}^{n} \left(\frac{1}{\tau_j} - \frac{1}{\tau_{j-1}} \right) - 2b \sum_{j=1}^{n} \left(\frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}} \right)$$

$$= \sum_{j=1}^{n} \left[\frac{x_j^2 - 2x_j x_{j-1} + x_{j-1}^2}{t_j - t_{j-1}} \right] + \sum_{j=1}^{n} \left(\frac{x_j^2}{\tau_j} - \frac{x_{j-1}^2}{\tau_{j-1}} \right)$$

$$+ b^2 \sum_{j=1}^{n} \left(\frac{1}{\tau_j} - \frac{1}{\tau_{j-1}} \right) - 2b \sum_{j=1}^{n} \left(\frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}} \right)$$

$$= \sum_{j=1}^{n} \left[\frac{x_j^2 - 2x_j x_{j-1} + x_{j-1}^2}{t_j - t_{j-1}} \right] + \sum_{j=1}^{n} \left(\frac{x_j^2}{\tau_j} - \frac{x_{j-1}^2}{\tau_{j-1}} \right)$$

$$+ b^2 \sum_{j=1}^{n} \left(\frac{1}{\tau_j} - \frac{1}{\tau_{j-1}} \right) - 2b \sum_{j=1}^{n} \left(\frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}} \right)$$

$$= \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} + \frac{x_n^2}{T - t_n} - \frac{a^2}{T} + b^2 \left(\frac{1}{T - t_n} - \frac{1}{T} \right)$$

$$-2b \left(\frac{x_n}{T - t_n} - \frac{a}{T} \right)$$

$\tau_j = T - t_j$ $x_0 = a$

$$\sum_{j=1}^{n} \left(\frac{x_j^2}{\tau_j} - \frac{x_{j-1}^2}{\tau_{j-1}} \right) = \left(\frac{x_1^2}{\tau_1} - \frac{x_0^2}{\tau_0}\right) + \left(\frac{x_2^2}{\tau_2} - \frac{x_1^2}{\tau_1}\right) + \ldots + \left(\frac{x_n^2}{\tau_n} - \frac{x_{n-1}^2}{\tau_{n-1}}\right) = \left(\frac{x_n^2}{\tau_n} - \frac{x_0^2}{\tau_0}\right)$$

$$\sum_{j=1}^{n} \left(\frac{1}{\tau_j} - \frac{1}{\tau_{j-1}} \right) = \left(\frac{1}{\tau_1} - \frac{1}{\tau_0}\right) + \left(\frac{1}{\tau_2} - \frac{1}{\tau_1}\right) + \ldots + \left(\frac{1}{\tau_n} - \frac{1}{\tau_{n-1}}\right) = \left(\frac{1}{\tau_n} - \frac{1}{\tau_0}\right)$$

$$\sum_{j=1}^{n} \left(\frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}} \right) = \left(\frac{x_1}{\tau_1} - \frac{x_0}{\tau_0}\right) + \left(\frac{x_2}{\tau_2} - \frac{x_1}{\tau_1}\right) + \ldots + \left(\frac{x_n}{\tau_n} - \frac{x_{n-1}}{\tau_{n-1}}\right) = \left(\frac{x_n}{\tau_n} - \frac{x_0}{\tau_0}\right)$$

$$= \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} + \frac{x_n^2}{T - t_n} - \frac{a^2}{T} + b^2 \left(\frac{1}{T - t_n} - \frac{1}{T} \right)$$

$$-2b \left(\frac{x_n}{T - t_n} - \frac{a}{T} \right)$$

$$= \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} + \frac{(b - x_n)^2}{T - t_n} - \frac{(b-a)^2}{T}.$$

$$\exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{\left(z_j - \frac{b(t_j - t_j - 1)}{\tau_j \tau_{j-1}} \right)^2}{\frac{t_j - t_{j-1}}{\tau_j \tau_{j-1}}} \right\}$$

$$= \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} - \frac{(b - x_n)^2}{2(T - t_n)} + \frac{(b-a)^2}{2T} \right\}.$$

- To change a density, we also need to account for the Jacobian of the change of variables. In this case, we have

$$z_j = \frac{x_j}{\tau_j} - \frac{x_{j-1}}{\tau_{j-1}}$$

$$\frac{\partial z_j}{\partial x_j} = \frac{1}{\tau_j}\ , \qquad j = 1, \ldots, n,$$

$$\frac{\partial z_j}{\partial x_{j-1}} = -\frac{1}{\tau_{j-1}}\ , \qquad j = 2, \ldots, n,$$

and all other partial derivatives are zero. This leads to the Jacobian matrix

$$J = \begin{bmatrix} \frac{1}{\tau_1} & 0 & \cdots & 0 \\ -\frac{1}{\tau_1} & \frac{1}{\tau_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{\tau_n} \end{bmatrix}$$

- Whose determinant is $\prod_{j=1}^{n} \frac{1}{\tau_j}$. Multiplying $f_{Z(t_1),\dots,Z(t_n)}(z_1,\dots,z_n)$ by this determinant and using the change of variables worked out above, we obtain the density for $X^{a\to b}(t_1),\cdots,X^{a\to b}(t_n)$,

$$J = \begin{bmatrix} \frac{1}{\tau_1} & 0 & \cdots & 0 \\ -\frac{1}{\tau_1} & \frac{1}{\tau_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{\tau_n} \end{bmatrix}$$

determinant is $\prod_{j=1}^{n} \frac{1}{\tau_j}$

$$f_{X^{a\to b}(t_1),\ldots,X^{a\to b}(t_n)}(x_1,\ldots,x_n)$$

$$= \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \sqrt{\frac{\tau_{j-1}}{\tau_j}}$$

$$\cdot \exp\left\{ -\frac{1}{2}\sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} - \frac{(b - x_n)^2}{2(T - t_n)} + \frac{(b-a)^2}{2T} \right\}$$

$$f_{Z(t_1),\ldots,Z(t_n)}(z_1,\ldots,z_n) = \exp\left\{ -\frac{1}{2}\sum_{j=1}^{n} \frac{\left(z_j - \frac{b(t_j - t_{j-1})}{\tau_j \tau_{j-1}} \right)^2}{\frac{t_j - t_{j-1}}{\tau_j \tau_{j-1}}} \right\} \cdot \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi \frac{t_j - t_{j-1}}{\tau_j \tau_{j-1}}}}$$

$$\exp\left\{ -\frac{1}{2}\sum_{j=1}^{n} \frac{\left(z_j - \frac{b(t_j - t_j - 1)}{\tau_j \tau_{j-1}} \right)^2}{\frac{t_j - t_{j-1}}{\tau_j \tau_{j-1}}} \right\}$$

$$= \exp\left\{ -\frac{1}{2}\sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} - \frac{(b - x_n)^2}{2(T - t_n)} + \frac{(b-a)^2}{2T} \right\}.$$

$$= \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \sqrt{\frac{\tau_{j-1}}{\tau_j}}$$

$$\cdot \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} - \frac{(b - x_n)^2}{2(T - t_n)} + \frac{(b-a)^2}{2T} \right\}$$

$$= \sqrt{\frac{T}{T - t_n}} \cdot \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}}$$

$\tau_j = T - t_j$

$$\cdot \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} - \frac{(b - x_n)^2}{2(T - t_n)} + \frac{(b-a)^2}{2T} \right\}$$

$$\prod_{j=1}^{n} \sqrt{\frac{\tau_{j-1}}{\tau_j}} = \sqrt{\frac{\tau_0}{\tau_1}} \times \sqrt{\frac{\tau_1}{\tau_2}} \times \ldots \times \sqrt{\frac{\tau_{n-1}}{\tau_n}} = \sqrt{\frac{\tau_0}{\tau_n}}$$

$$= \sqrt{\frac{T}{T - t_n}} \cdot \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \times \sqrt{\frac{2\pi}{2\pi}}$$

$$\cdot \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} - \frac{(b - x_n)^2}{2(T - t_n)} + \frac{(b-a)^2}{2T} \right\}$$

$$= \frac{p(T - t_n, x_n, b)}{p(T, a, b)} \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j), \tag{4.7.6}$$

where

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left\{ -\frac{(y-x)^2}{2\tau} \right\}$$

is the transition density for Brownian motion. 課本p.108

4.7.5 Brownian Bridge as a Conditioned Brownian Motion

- The joint density (4.7.6) for $X^{a\to b}(t_1),\cdots,X^{a\to b}(t_n)$ permits us to give one more interpretation for Brownian bridge from a to b on $[0,T]$.
  It is a Brownian motion $W(t)$ on this time interval, starting at $W(0)=a$ and conditioned to arrive at b at time T (i.e., conditioned on $W(T)=b$).
  Let $0=t_0<t_1<t_2<\ldots<t_n<T$ be given. The joint density of $W(t_1),\ldots,W(t_n),W(T)$ is

$$f_{W(t_1),\ldots,W(t_n),W(T)}(x_1,\ldots,x_n,b)=p(T-t_n,x_n,b)\prod_{j=1}^{n}p(t_j-t_{j-1},x_{j-1},x_j)$$

$$,\quad where\ W(0)=x_0=a \qquad \textbf{(4.7.7)}$$

This is because $p(t_1-t_0,x_0,x_1)=p(t_1,a,x_1)$ is the density for the Brownian motion going from $W(0)=a$ to $W(t_1)=x_1$ in the time between $t=0$ and $t=t_1$. Similarly, $p(t_2-t_1,x_1,x_2)$ is the density for going from $W(t_1)=x_1$ to $W(t_2)=x_2$ between time $t=t_1$ and $t=t_2$. The joint density for $W(t_1)$ and $W(t_2)$ is then the product

$$p(t_1,a,x_1)\,p(t_2-t_1,x_1,x_2).$$

Continuing in this way, we obtain the joint density (4.7.7).

The marginal density of $W(T)$ is $p(T, a, b)$.

So, the density of $W(t_1),\ldots,W(t_n)$ conditioned on $W(T)=b$ is thus the quotient

$$\frac{p(T - t_n, x_n, b)}{p(T, a, b)} \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j)$$

marginal density of W(T)

joint density of $W(t_1), \ldots, W(t_n), W(T)$

and this is $f_{X^{a\to b}(t_1),\ldots,X^{a\to b}(t_n)}(x_1, \ldots, x_n)$ of (4.7.6).

Finally, let us define

$$M^{a\to b}(T) = \max_{0\le t\le T} X^{a\to b}(t)$$

to be the maximum value obtained by the Brownian bridge from a to b on $[0,T]$. This random variable has the following distribution.

Corollary 4.7.7.

The density of $M^{a\to b}(T)$ is

Brownian bridge from a to b on [0, T] ⟶ a Brownian motion W(t) on this time interval, starting at W(0) = a and conditioned on W(T) = b

$$f_{M^{a\to b}(T)}(y) = \frac{2(2y-b-a)}{T} e^{-\frac{2}{T}(y-a)(y-b)} \quad , \quad y > \max\{a,b\}. \qquad (4.7.8)$$

Proof : Because the Brownian bridge from 0 to w on $[0,\mathrm{T}]$ is a Brownian motion conditioned on $W(T)=w$, the maximum of $X^{0\to w}$ on $[0,\mathrm{T}]$ is the maximum of W on $[0,\mathrm{T}]$ conditioned on $W(T)=w$. Therefore, the density of $M^{0\to w}(T)$ was computed in Corollary 3.7.4 and is

$$f_{M^{0\to w}(T)}(m) = \frac{2(2m-w)}{T} e^{-\frac{2m(m-w)}{T}} \quad , \quad w < m \ , \ m > 0. \qquad (4.7.9)$$

Corollary 3.7.4. *The conditional distribution of $M(t)$ given $W(t)=w$ is*

the maximum of the Brownian motion on [0, t]

$$f_{M(t)|W(t)}(m|w) = \frac{2(2m-w)}{t} e^{-\frac{2m(m-w)}{t}}, \ w \le m, m > 0.$$

Corollary 4.7.7.

The density of $M^{a \to b}(T)$ is

$$f_{M^{a \to b}(T)}(y) = \frac{2(2y - b - a)}{T} e^{-\frac{2}{T}(y-a)(y-b)} \quad , \qquad y > \max\{a, b\}. \qquad (4.7.8)$$

Proof : Because the Brownian bridge from 0 to w on $[0,\mathrm{T}]$ is a Brownian motion conditioned on $W(T) = w$, the maximum of $X^{0 \to w}$ on $[0,\mathrm{T}]$ is the maximum of W on $[0,\mathrm{T}]$ conditioned on $W(T) = w$. Therefore, the density of $M^{0 \to w}(T)$ was computed in Corollary 3.7.4 and is

$$f_{M^{0 \to w}(T)}(m) = \frac{2(2m - w)}{T} e^{-\frac{2m(m-w)}{T}} \quad , \qquad w < m \ , \ m > 0. \qquad (4.7.9)$$

The density of $f_{M^{a \to b}(T)}(y)$ can be obtained by translating from the initial condition $W(0) = a$ to $W(0) = 0$ and using (4.7.9). In particular, in (4.7.9) we replace m by $y - a$ and replace w by $b - a$. This result in (4.7.8).